**Given:** A graph $G = (V, E)$, edges weighted by $c$, where $c$ satisfies triangle inequality

$$c(x, y) + c(y, z) \geq c(x, z)$$

and $V$ is partitioned as

$$V = R \cup S$$

($R =$ required vertices, *terminals*; $S =$ *Steiner vertices*),

**Find:** A minimum-weight tree that spans all of $R$ (and possibly some of $S$).
Triangle inequality implies $G$ is complete (if $G$ is not complete, we may still make it so for the purpose of this problem by using instead its metric completion: for every pair of vertices $u, \nu$, add an edge of weight $d(u, \nu)$, where $d$ is the length of the shortest path between $u$ and $\nu$.

NP-hard optimization problem

Problem goes back to Gauss (early 19th century)

Often considered in the Euclidean plane; then, $R$ only given, $S$ implicitly all the vertices of the plane
Note: sometimes the Steiner points are actually necessary.
Since $G$ is complete, we can try to ignore the Steiner points:

**Algorithm:** Find a minimum spanning tree on $R$, use this as an “approximate” Steiner tree.

**How good (or bad) is this?**
Double every edge of the tree.

(Note: the image shows Steiner vertices in the tree; this may happen if, as mentioned above, we had to form the metric completion of the original input graph.)

This graph has all even degrees; find an Eulerian tour (possible in time $O(n)$).

The cost of the tour: $\leq 2 \cdot \text{OPT}$.

Now find a Hamilton tour of $R$ by shortcutting around vertices of $S$ in the tree and previously visited vertices of $R$. 
The cost of the Euler tour $\leq 2 \cdot \text{OPT}$.

Shortcuts around previously seen vertices and Steiner points can only decrease the cost because of triangle inequality, thus the cost of the Hamilton tour is still at most $2 \cdot \text{OPT}$. Removing the longest edge of the Hamilton tour gives a Steiner tree of cost at most $2 \cdot \text{OPT}$ (actually, at most $2(1 - 1/k) \cdot \text{OPT}$, where $k = |R|$...).
The analysis is actually tight. (Example?)

More complicated algorithms can improve the approximation guarantee to $11/6$ (instead of 2) [Zelikovsky]

Best known guarantees:

- for the general graph problem: roughly $5/3$ ($3/2$ is the goal, but seems unreachable right now)
- if the terminals are in the Euclidean plane, $1 + \epsilon$ is possible for any $\epsilon > 0$ (we'll see this later in the course)
Given: A graph $G = (V, E)$, edges weighted by $c$, where $c$ satisfies triangle inequality

$$c(x, y) + c(y, z) \geq c(x, z)$$

Find: A minimum-weight cycle that visits every vertex exactly once.

Note:

- Solution exists because $G$ is wlog complete
- If no triangle inequality, no approximation ratio is possible unless P=NP. (Proof sketch on the board.)
Spanning tree heuristic

1. Find an MST $T$ of $G$.
2. Double every edge to get an Eulerian graph.
3. Find an Euler tour $\mathcal{T}$ on this new graph.
4. Output the tour that visits vertices of $G$ in order of their first appearance in $\mathcal{T}$. Let $\mathcal{C}$ be this tour.

Claim: The above is a 2-approximation algorithm for metric TSP.
MST approximation to TSP: analysis

The same argument as for the Steiner tree problem:

1. \( \text{cost}(T) \leq \text{OPT} \) (Take an optimal TSP tour, remove an edge, giving an MST of smaller cost.)
2. \( \text{cost}(T) \leq 2\text{OPT} \)
3. Shortcutting gives \( \text{cost}(C) \leq 2\text{OPT} \).

Tight example?
Where do we lose factors in the analysis?

Answer: Making an Euler tour from $T$ takes a factor of 2 immediately.

Note: ANY Eulerian graph built on top of $T$ would work!

So: Can we find a cheaper Eulerian graph containing $T$?
Eulerian condition: every degree is even.

So: find a minimum-weight graph that increases by one the degree of every odd-degree vertex.

In other words, this is called?

Find a minimum-weight matching on the set of odd-degree vertices of $T$!
Christofides’ algorithm

1. Find an MST $T$ of $G$.
2. Find a minimum-weight matching on the set of odd degree vertices of $T$, and add the matching edges to $T$.
3. Find an Euler tour $\mathcal{T}$ on this new graph.
4. Output the tour that visits vertices of $G$ in order of their first appearance in $\mathcal{T}$. Let $C$ be this tour.

**Claim:** The above is a $3/2$-approximation algorithm for metric TSP.
Christofides’ approximation to TSP: analysis

The same argument as before:

1. \( \text{cost}(T) \leq \text{OPT} \) (Take an optimal TSP tour, remove an edge, giving an MST of smaller cost.)

2. \( \text{cost}(T) \leq \frac{3}{2} \text{OPT} \)

3. Shortcutting gives \( \text{cost}(C) \leq \frac{3}{2} \text{OPT} \).

The missing piece: The minimum-weight perfect matching on the set of odd-degree vertices of \( T \) has weight \( \leq \text{OPT}/2 \).

Argument: on the board.