Steiner tree problem: definition

Given: A graph $G = (V, E)$, edges weighted by c, where c satisfies triangle inequality

$$c(x, y) + c(y, z) \geq c(x, z)$$

and V is partitioned as

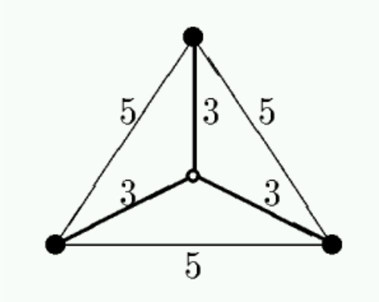
$$V = R \cup S$$

(R = required vertices, *terminals*; S = *Steiner vertices*),
Find: A minimum-weight tree that spans all of R (and possibly some of S).

Steiner tree problem: notes

- Triangle inequality implies G is complete (if G is not complete, we may still make it so for the purpose of this problem by using instead its metric completion: for every pair of vertices u, v, add an edge of weight $d(u, v)$, where d is the length of the shortest path between u and v.
- NP-hard optimization problem
- Problem goes back to Gauss (early 19th century)
- Often considered in the Euclidean plane; then, R only given, S implicitly all the vertices of the plane

Steiner tree: simple example



Note: sometimes the Steiner points are actually necessary.

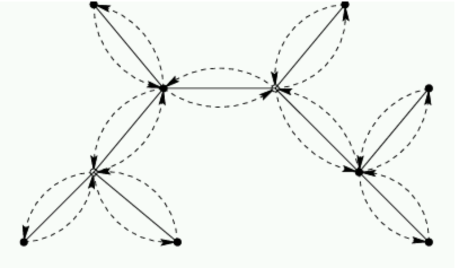
Minimum spanning tree heuristic

Since G is complete, we can try to ignore the Steiner points:
Algorithm: Find a minimum spanning tree on R, use this as an "approximate" Steiner tree.

How good (or bad) is this?

Minimum spanning tree analysis

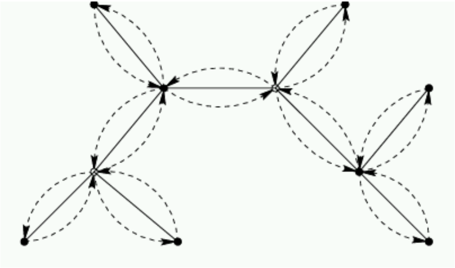
1. Double every edge of the tree.



(Note: the image shows Steiner vertices in the tree; this may happen if, as mentioned above, we had to form the metric completion of the original input graph.)

2. This graph has all even degrees; find an Eulerian tour (possible in time $O(n)$).
3. The cost of the tour: $\leq 2 \cdot \mathrm{OPT}$.
4. Now find a Hamilton tour of R by shortcutting around vertices of S in the tree and previously visited vertices of R

Minimum spanning tree analysis (2)



The cost of the Euler tour $\leq 2 \cdot \mathrm{OPT}$.

Shortcuts around previously seen vertices and Steiner points can only decrease the cost because of triangle inequality, thus the cost of the Hamilton tour is still at most $2 \cdot$ OPT.
Removing the longest edge of the Hamilton tour gives a Steiner tree of cost at most $2 \cdot \mathrm{OPT}$ (actually, at most $2(1 - 1/k) \cdot \mathrm{OPT}$, where $k = |R|$...)

Notes

- The analysis is actually tight. (Example?)
- More complicated algorithms can improve the approximation guarantee to $11/6$ (instead of 2) [Zelikovsky]
- Best known guarantees:
  - for the general graph problem: roughly $5/3$ ($3/2$ is the goal, but seems unreachable right now)
  - if the terminals are in the Euclidean plane, $1 + \epsilon$ is possible for any $\epsilon > 0$ (we'll see this later in the course)

(Metric) Traveling salesman problem (TSP)

Given: A graph $G = (V, E)$, edges weighted by c, where c satisfies triangle inequality

$$c(x, y) + c(y, z) \geq c(x, z)$$

Find: A minimum-weight cycle that visits every vertex exactly once.

Note:

- Solution exists because G is wlog complete
- If no triangle inequality, no approximation ratio is possible unless P=NP. (Proof sketch on the board.)

Spanning tree heuristic

1. Find an MST T of G.
2. Double every edge to get an Eulerian graph.
3. Find an Euler tour $\mathcal{T}$ on this new graph.
4. Output the tour that visits vertices of G in order of their first appearance in $\mathcal{T}$. Let $\mathcal{C}$ be this tour.

Claim: The above is a 2-approximation algorithm for metric TSP.

MST approximation to TSP: analysis

The same argument as for the Steiner tree problem:

1. $cost(T) \leq$OPT (Take an optimal TSP tour, remove an edge, giving an MST of smaller cost.)
2. $cost(\mathcal{T}) \leq 2$OPT
3. Shortcutting gives $cost(\mathcal{C}) \leq 2$OPT.

Tight example?

3/2 approximation for metric TSP

Where do we lose factors in the analysis?

Answer: Making an Euler tour from T takes a factor of 2 immediately.

Note: **ANY** Eulerian graph built on top of T would work!

So: Can we find a cheaper Eulerian graph containing T?

3/2 approximation for metric TSP (2)

Eulerian condition: every degree is even.

So: find a minimum-weight graph that increases by one the degree of every odd-degree vertex.

In other words, this is called?

Find a minimum-weight **matching** on the set of odd-degree vertices of T!

Christofides' algorithm

1. Find an MST T of G.
2. Find a minimum-weight matching on the set of odd degree vertices of T, and add the matching edges to T.
3. Find an Euler tour $\mathcal{T}$ on this new graph.
4. Output the tour that visits vertices of G in order of their first appearance in $\mathcal{T}$. Let $\mathcal{C}$ be this tour.

Claim: The above is a 3/2-approximation algorithm for metric TSP.

Christofides' approximation to TSP: analysis

The same argument as before:

1. $cost(T) \leq$OPT (Take an optimal TSP tour, remove an edge, giving an MST of smaller cost.)
2. $cost(\mathcal{T}) \leq 3/2$OPT
3. Shortcutting gives $cost(\mathcal{C}) \leq 3/2$OPT.

The missing piece: The minimum-weight perfect matching on the set of odd-degree vertices of T has weight $\leq$OPT$/2$.
Argument: on the board.