Homework 1 — CSE 591 Fall 2008

Due September 9

Exercises (solve for practice)

1 Let X be a random variable taking on values $0, 1, 2, \ldots,$. The expectation of X is defined as $\mathbf{E}[X] = \sum_i i \cdot \mathbf{P}[X = i]$. Show that $\mathbf{E}[X] = \sum_i \mathbf{P}[X \geq i]$.

2 Suppose X and Y are two random variables. Then $\mathbf{E}[X + Y] = \mathbf{E}[X] + \mathbf{E}[Y]$. If X and Y are also independent, then $\mathbf{E}[XY] = \mathbf{E}[X] \cdot \mathbf{E}[Y]$.

3 Suppose that 5 men out of 10000 and 25 women out of 100000 are colorblind. A colorblind person is chosen at random. What is the probablity that the person is male? (Assume that there are as many males as females in the general population.)

4 *Banach's matchbox problem* (attributed to H. Steinhaus.) A certain mathematician always carries a matchbox in his right pocket and another in his left. When he needs a match, he selects a pocket at random. Suppose that initially, each box contained N matches. What is the probability that when he first reaches for an empty matchbox, the other one contains r matches?

Problems (hand in solutions for class credit)

5 **(a)** Suppose you are given a coin for which the probability of heads, say p, is unknown. How can you use this coin to generate unbiased coin-flips? Give a scheme for which the expected number of flips of the biased coin for extracting one unbiased coin-flip is no more than $1/(p(1-p))$.
(b) Suppose you are given a two-sided fair coin (each side comes up with probability 1/2 when tossed). Describe a method to generate a random number from $\{1, 2, 3\}$ with a uniform distribution. Your method should have constant expected running time. Is it possible to do the same but with a guaranteed bound on the running time (even if not constant)?

6 **Matching nuts and bolts.** Suppose we are given two sets A and B containing n elements each. Both sets can be ordered linearly, but we cannot directly compare two elements of A or two elements of B. Instead, the primitive operation is comparing an element $a \in A$ to an element $b \in B$. This comparison has three possible outcomes:
(1) $a < b$, meaning that the index of a in the sorted set A is smaller than the index of b in the sorted set B,
(2) $a = b$, meaning that the indices of a and b are equal, and
(3) $a > b$, meaning that the index of a is greater than the index of b.

Assume it is known that both sets can be ordered linearly, say as $a_1 < a_2 < \cdots < a_n$ and $b_1 < b_2 < \cdots < b_n$, so that for every $i < j$, we have $a_i < b_j$ and $b_i < a_j$, where $<$ agrees with the outcome of the comparison operation described above..

Give a randomized algorithm that sorts both A and B in expected time $O(n \log n)$. Give a rigorous analysis of the running time. Can you find a deterministic algorithm with the same bound?

7 Consider the min-cut algorithm we discussed in class.
(a) Study the proof that the algorithm finds a minimum cut with probability at least $2/n^2$ and prove that there can be at most $O(n^2)$ different minimum cuts in the graph.

Now suppose the goal, instead of finding a minimum cut in the graph, was to find a minimum cut separating the vertices s and t. Given a graph with two distinguished vertices s and t, an s-t cut is a set of edges whose removal disconnects s from t. The goal is to find an s-t cut of minimum cardinality. As the algorithm proceeds, the vertex s may be joined with other vertices; call the resulting vertex the s-vertex; similarly with t.

(b) Show that there are graphs in which the probability that the algorithm finds an s-t minimum cut is exponentially small.

(c) What is the maximum possible number of different s-t minimum cuts in a graph on n vertices?

8

(a) Describe a method for using a fair coin (a source of unbiased random bits) to generate a random permutation of $\{1, 2, \ldots, n\}$. Your method should use an expected $O(n \log n)$ random bits and take expected time $O(n \log n)$. (Hint: use a "quicksort" paradigm.)
(b) Describe a method for using a source of $O(\log n)$-bit random numbers to generate a random permutation of $\{1, 2, \ldots, n\}$. Your method should use an expected $O(n)$ random numbers and take expected time $O(n)$, assuming you can perform operations on these numbers in unit time.

9

(Due to M. Rabin.) Consider the following game. A friend writes down two numbers on two slips of paper and then randomly puts one in one hand and the other in the other hand. You get to pick a hand and see the number in it. You then can either keep the number you saw or else return it and get the other number. Say you end up with the number x and the other number was y. Then, your gain is $x - y$.

For a given (possibly) randomized stratedy S, let $\mathbf{E}_{x,y}(S)$ denote its expected gain, given that the two numbers are x and y. For instance, if S is a deterministic strategy, then $\mathbf{E}_{x,y}(S) = 1/2\cdot$ (gain of S given that it is initially shown x and the other number is y)$+1/2\cdot$ (gain of S given that it is initially shown y and the other number is x).

(a) Consider the strategy $S =$"if the first number I see is ≥ -17, then I keep it, else I switch". What is $\mathbf{E}_{x,y}(S)$ in terms of x and y?

(b) Give a randomized strategy S such that $\mathbf{E}_{x,y}(S) > 0$ for all $x \neq y$.

(c) Is it possible to achieve the same (always have a positive gain, regardless of the values x and y) with a deterministic strategy? Justify your answer.

10

(Due to E. Lauer) Suppose that in the above game, your friend picks his two numbers by choosing the pair $\{3^{i-1}, 3^i$ with probability $1/2^i$ (for $i \geq 1$). In class, we saw that it seems that no matter what number you see, you should switch since for any x, given that you see x, your expected gain of switching is greater than your gain of not switching. Of course, this is weird, since the "always switch" strategy is equivalent to "always keep" strategy.

What is going on here? Try to give as satisfying an explanation as you can. Also, discuss the performance of the "switch" strategy in terms of random variables X_i that correspond to its gain given that you see 3^i initially.