Due: Tuesday, 1/23/2006 before class

1. Consider the alphabet consisting of all length-three binary vectors:
$$\Sigma = \left\{ \begin{pmatrix} 0\\0\\0 \end{pmatrix}, \begin{pmatrix} 0\\0\\1 \end{pmatrix}, \begin{pmatrix} 0\\1\\0 \end{pmatrix}, \begin{pmatrix} 0\\1\\1 \end{pmatrix}, \begin{pmatrix} 1\\0\\0 \end{pmatrix}, \begin{pmatrix} 1\\0\\1 \end{pmatrix}, \begin{pmatrix} 1\\1\\0 \end{pmatrix}, \begin{pmatrix} 1\\1\\1 \end{pmatrix} \right\}.$$
Think of a string over Σ as three binary numbers, one in each row, left to right. For example, the string $u = \begin{pmatrix} 0\\0\\1 \end{pmatrix}\begin{pmatrix} 0\\1\\0 \end{pmatrix}\begin{pmatrix} 1\\1\\1 \end{pmatrix}\begin{pmatrix} 1\\1\\0 \end{pmatrix}$ corresponds to the numbers 0011 (top row), 0111 (middle row) and 1010 (bottom row). A string represents a correct binary addition if the sum of the first two numbers equals the third: in this example, it is true that (in binary) $0011 + 0111 = 1010$.
Let L be the set of all strings over Σ that represent correct binary additions. For example, the string v above belongs to L. However, the string $\begin{pmatrix} 0\\0\\1 \end{pmatrix}\begin{pmatrix} 1\\1\\1 \end{pmatrix}\begin{pmatrix} 1\\0\\0 \end{pmatrix}$ does not belong to L because $011 + 010 \neq 110$.
Draw the state diagram of a **deterministic** finite automaton that accepts L. The input must be processed left-to-right, that is, the most significant bits must be read first.

2. If A is any language, let $A_{\frac{1}{2}-}$ be the set of all first halves of strings in A, so that
$$A_{\frac{1}{2}-} = \{x \mid \text{for some } y,\ |x| = |y| \text{ and } xy \in A\}.$$
Show that if A is regular, then so is $A_{\frac{1}{2}-}$.

3. Let $\Sigma = \{0, 1\}$ and let
$$D = \{w \mid w \text{ contains an equal number of occurrences of strings 01 and 10}\}.$$
Thus $101 \in D$ because 101 contains a single 01 and a single 10, but $1010 \notin D$ because 1010 contains two 10s and one 01. Show that D is a regular language.

4. Let $B = \{1^k y \mid y \in \{0, 1\}^* \text{ and } y \text{ contains at least } k \text{ 1s, for } k \geqslant 1\}$. Show that B is regular.

5. Let $M = (Q, \Sigma, \delta, q_0, F)$ be a DFA and let h be a state of M called its "home". A *synchronizing sequence* for M and h is a string $s \in \Sigma^*$ where $\delta(q, s) = h$ for every $q \in Q$. (Here we have extended δ to strings so that $\delta(q, s)$ equals the state where M ends up when M starts at state q and reads input s.) Say that M is *synchronizable* if it has a synchronizing sequence for some state h. Prove that, if M is a k-state DFA, with a synchronizing sequence, then it has a synchronizing sequence of length at most k^3. Can you improve upon this bound?