Homework 1 CSE 550 Fall 2007 Arizona State University

Due: Tuesday 8/28 before 10:30

1. Suppose the graph G on n vertices is complete: it contains all the possible edges. How many different spanning trees does it have? To answer this, study the following algorithm:

   Input: spanning tree T on $V = \{v_1, v_2, \ldots, v_n\}$
   Output: sequence $(a_1, a_2, \ldots, a_{n-2})$
   (1) Let v be the leaf of T with the smallest index.
   (2) Let a_1 be the index of v's neighbor in T.
   (3) Repeat steps (1) and (2) with $T - v$ and $(a_2, \ldots, a_{n-2})$.

2. Adding vertices. Suppose that, after you have solved a shortest-path problem, a new vertex v_{n+1} is added to the graph. Describe how to efficiently (without recomputing the shortest-path tree) update shortest-path distances from the origin vertex s to every other vertex if
   (a) all edge lengths are nonnegative and vertex v_{n+1} has only incoming edges,
   (b) all edge lengths are nonnegative and vertex v_{n+1} has incoming as well as outgoing edges, and
   (c) edge lengths are arbitrary, but vertex v_{n+1} has only incoming edges.

3. We call an edge e MST-critical if deleting e from the graph increases the cost of the minimum spanning tree. The MST-most-critical edge is one whose deletion increases the cost of the minimum spanning tree by the maximum amount.
   (a) Does a graph always contain an MST-critical edge?
   (b) Suppose a graph does contain an MST-critical edge. Describe an $O(nm)$ algorithm that finds the MST-most-critical edge. Can you develop an even faster algorithm?

4. Give an algorithm to find the longest path in a directed acyclic graph. How fast can you make the algorithm?

5. Suppose the edge-weights are integers from $\{0, 1, \ldots, C\}$ for some constant C. Show how to implement Dijkstra's algorithm in time $O(m)$, where m is the number of edges. (Hint: use an array indexed by $0, 1, \ldots, mC$ to store the vertices according to their current distance estimates.)

6. Let $G = (V, E)$ be a directed graph with nonnegative edge-weight function w, and $s, t \in V$. Consider the problem of maximizing $\ell(t)$ subject to (1) $\ell(s) = 0$, (2) $\ell(v) \leq \ell(u) + w(u, v)$ for all $(u, v) \in E$. What is another name for $\ell(t)$? Prove that your answer is correct.